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THE DISPLACEMENTS IN AN ELASTIC HALF-SPACE WHEN A LOAD MOVES ALONG A BEAM LYING ON ITS SURFACE†

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The motion, with constant velocity, of a normal load along an elastic beam lying on an elastic isotropic homogeneous half-space is considered. A method for the approximate calculation of the normal displacements of the surface of the half-space for subsonic velocities of motion is developed. An estimate is given of the expressions obtained and a comparison is made with existing results for the problem of the motion of a point load along a half-space.

The problem of the motion of a point load along an infinitely long beam lying on an elastic half-space was considered in [1]. In addition to the formulation of this problem, an axial compression of the beam was introduced in [2]. The unsteady problem of the action of a uniformly moving force on a homogeneous isotropic half-space, taking into account the sudden application of the load, was solved in [3]. The equivariable motion of a force along a Timoshenko-type beam lying on an elastic base was investigated in [4]. The stress state of an elastic half-space due to a uniformly moving normal load, distributed in a strip of the surface of half-space, was considered in [5]. A method of determining the displacements in an elastic half-space containing a depressed cylindrical cavity when an oscillating point load along the generatrix of the cylinder moves uniformly over the surface of a half-space was presented in [6]. The deformation of an elastic beam lying on a Winkler base when there is non-axial bending of a mobile point load was considered in [7]. A solution of the problem of the motion, with constant velocity, of a point load over the surface of an elastic half-space was given in [8].

A comparison of the results obtained in the above-mentioned papers showed that, in the limiting case, when the materials of the half-space and the beam are the same, the approximate solution presented converges asymptotically to the solution obtained in [8] as one moves away from the point of application of the load.

1. FORMULATION OF THE PROBLEM

A point load of intensity P moves with constant velocity c (see Fig. 1) along a beam lying on an elastic half-space. The oscillations of the axis of the elastic beam are described by the equation [1]

$$B\frac{\partial^4 w}{\partial x^4}+\rho_b\frac{\partial^2 w}{\partial t^2}=p(x,t) \tag{1.1}$$

where $w(x, t)$ is the normal displacement of the beam axis, $B = E_bJ$ is its bending stiffness, ρ_b is the density of the beam material and $p(x, t)$ is the intensity of the load applied to the beam.

In a fixed system of coordinates, the displacement vector in an elastic half-space satisfies the equation [9]

$$\mu\Delta\mathbf{u}+(\lambda+\mu)\nabla\operatorname{div}\mathbf{u}=\rho\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{1.2}$$

where $\mathbf{u}(u_x, u_y, u_z)$ is the displacement vector and λ, μ, ρ are the constants of the material of the base. It is assumed that no friction force acts between the beam and the surface of the half-space, i.e.

$$\frac{\partial u_z}{\partial x}+\frac{\partial u_x}{\partial z}=0,\quad \frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}=0 \quad \text{when } z=0 \tag{1.3}$$

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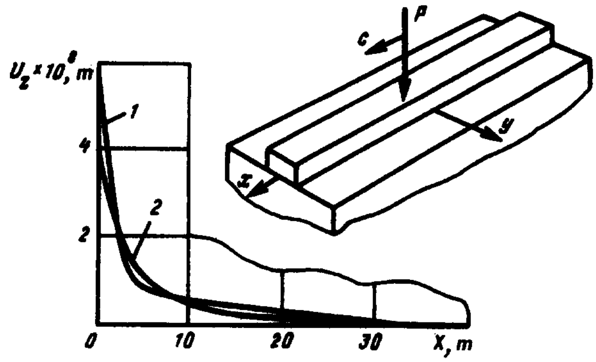

Fig. 1.

To solve the problem we will use the condition that the normal displacements of the axis of the beam and the elastic half-space under it are identical, namely

$$w(x, t) = u_z(x, y = 0, z = 0, t) \tag{1.4}$$

The load is applied to the base uniformly over the width of the supporting strip.

The problem is assumed to be stationary in a moving system of coordinates, in which the load is applied at the origin of coordinates.

The purpose of this investigation is to obtain an approximate expression for the normal displacement of the surface of the half-space, which can be effectively used in calculations.

We will introduce a moving system of coordinates, connected with the load [1]. In this system, the normal displacement of the axis of the elastic beam satisfies the equation

$$B\frac{\partial^4 W}{\partial x^4} + \rho_b c^2 \frac{\partial^2 W}{\partial t^2} = P(x) \tag{1.5}$$

Equation (1.2) in the moving system takes the form

$$\mu \Delta \mathbf{U} + (\lambda + \mu)\nabla \operatorname{div} \mathbf{U} = \rho c^2 \frac{\partial^2 \mathbf{U}}{\partial x^2} \tag{1.6}$$

Here $\mathbf{U}(U_x, U_y, U_z)$ is the displacement vector in the half-space in the moving system.

The displacement field $\mathbf{U}$ can be expanded in potential and solenoidal components: $\mathbf{U} = \nabla\Phi + \mathbf{U}'$. The potential function Φ and the vector $\mathbf{U}'$ satisfy the equations

$$\left(\Delta - h^2\frac{\partial^2}{\partial x^2}\right)\Phi = 0, \quad \left(\Delta - k^2\frac{\partial^2}{\partial x^2}\right)\mathbf{U}' = 0, \quad \operatorname{div}\mathbf{U}' = 0$$
$$h = \frac{c}{c_p}, \quad k = \frac{c}{c_s}, \quad c_p = \sqrt{\frac{2\mu + \lambda}{\rho}}, \quad c_s = \sqrt{\frac{\mu}{\rho}} \tag{1.7}$$

where h and k are the ratios of the velocity of motion of the load to the longitudinal and transverse velocities of sound in the beam.

We will apply a Fourier cosine and sine transformation to the first two equations of (1.7). The number of unknown functions (inverse transformants) is reduced to two when using (1.3) and the third equation of (1.7). The remaining two unknown functions are found from condition (1.4) that the displacement of the beam axis and the surface of the half-space under it should be identical.

We will write an expression for the normal displacement of the surface of the elastic half-space under the moving load [1]

$$U_z(0,0,0) = \frac{4(1-\nu^2)}{\pi^2 E}\frac{P}{b}\int_0^\infty \frac{S(u)du}{u + \varepsilon u^2(u^2 + \delta^2)S(u)} \tag{1.8}$$

where

$$S(u)=\frac{k^2}{1-\nu}\int_0^\infty\frac{D_1\sin(u\tau)d\tau}{\tau[4D_2(D_1-D_2)D_0^2-k^4]} \tag{1.9}$$

$$D_0^2=1+\tau^2,\quad D_1^2=1+\tau^2-h^2,\quad D_2^2=1+\tau^2-k^2$$

$$\varepsilon=\frac{2(1-\nu^2)}{\pi E}\left(\frac{2}{b}\right)^4 B,\quad \delta=\sqrt{\frac{\rho_b}{B}}c\left(\frac{b}{2}\right)$$

E and ν are the constants of elasticity of the base and b is the width of the supporting strip.

Following the method of solution proposed in [1], we will represent the normal displacement of the surface of the half-space in the form

$$U_z(x,y,0)=\frac{4(1-\nu^2)}{\pi^2E}\frac{P}{b}\int_0^\infty\frac{\alpha^2I(\alpha,y)}{1+\varepsilon\alpha(\alpha^2-\delta^2)S(\alpha)}\cos\left(\frac{2x}{b}\alpha\right)d\alpha \tag{1.10}$$

where

$$I(\alpha,y)=\frac{k^2}{1-\nu}\int_0^\infty\frac{d_1\sin\beta}{\beta[4d_2(d_1-d_2)d_0^2-\alpha^4k^4]}\cos\left(\frac{2y}{b}\beta\right)d\beta \tag{1.11}$$

$$d_0^2=\beta^2+\alpha^2,\quad d_1^2=\beta^2+(1-h^2)\alpha^2,\quad d_2^2=\beta^2+(1-k^2)\alpha^2$$

A considerable amount of calculations is required to calculate the displacements using Eq. (1.10), since the integrals contain rapidly oscillating functions. We show below how one can obtain an approximate expression for (1.10) with fairly generally assumptions.

2. THE APPROXIMATION OF THE INTEGRALS $I(\alpha, y)$ AND $S(\alpha)$ BY POWER SERIES

The majority of modern terrestrial transport vehicles move with velocities that are an order of magnitude smaller than the velocity of propagation of acoustic waves, and consequently, at subsonic velocities of motion of the load the parameters h^2 and k^2 are small.

We will expand the functions d_1 and d_2 in series in powers of h^2 and k^2 and take the first two terms of the expansions

$$d_1\approx d_0-\frac{\alpha^2}{2d_0}h^2,\quad d_2\approx d_0-\frac{\alpha^2}{2d_0}k^2 \tag{2.1}$$

In the expression under the integral $I(\alpha,y)$ we can write

$$\sin\beta\cos\left(\frac{2y}{b}\beta\right)=\frac{1}{2}(\sin\alpha_1\beta-\sin\alpha_2\beta),\quad \alpha_1=\frac{2y}{b}+1,\quad \alpha_2=\frac{2y}{b}-1 \tag{2.2}$$

We substitute expressions (2.1) and (2.2) into (1.11) and make the replacement $\beta=\alpha\tau$. At subsonic velocities it is possible to write

$$\frac{1}{D_0^2-\sigma}=\sum_{k=0}^\infty\frac{\sigma^k}{D_0^{2k+2}},\quad \sigma=\frac{2k^4-k^2h^2}{2(k^2-h^2)}$$

We finally have the following approximate expression

$$I(\alpha,y)\approx\frac{1}{4(k^2-h^2)}\frac{k^2}{1-\nu}\frac{1}{\alpha^3}\left[f_0(\alpha\alpha_1)-f_0(\alpha\alpha_2)+\xi\sum_{k=0}^\infty(f_{k+1}(\alpha\alpha_1)-f_{k+1}(\alpha\alpha_2))\sigma^k\right] \tag{2.3}$$

where

$$f_k(\alpha)=\int_0^\infty\frac{\sin\alpha\tau d\tau}{\tau(\tau^2+1)^{\frac{1}{2}+k}},\quad \xi=\frac{2k^4-2k^2h^2+h^4}{2(k^2-h^2)} \tag{2.4}$$

Similarly, using the approximations

$$D_1 \approx D_0 - \frac{1}{2D_0}h^2, \quad D_2 \approx D_0 - \frac{1}{2D_0}k^2 \tag{2.5}$$

integral (1.9) can be represented in the form

$$S(\alpha) \approx \frac{1}{2(k^2-h^2)}\frac{k^2}{1-\nu}\left\{f_0(\alpha)+\xi\sum_{k=0}^{\infty} f_{k+1}(\alpha)\sigma^k\right\} \tag{2.6}$$

It follows from (2.3) and (2.6) that the problem of obtaining an approximate expression for the normal displacement $U_z(x, y, 0)$ of the surface of the half-space has been reduced to evaluating the integrals (2.4).

3. EXPANSIONS OF THE FUNCTIONS $f_k(\alpha)$ IN POWER SERIES

The integral of $f_0(\alpha)$ can be expressed in terms of the MacDonald function

$$f_0(\alpha) = \int_0^{\alpha} K_0(u)du \tag{3.1}$$

Using the well-known expansion of $K_0(\alpha)$ in a power series [10], we have

$$f_0(\alpha) = \sum_{k=0}^{\infty} \frac{2}{(2k+1)(k!)^2}\left\{\sum_{n=1}^{k}\frac{1}{n} - C + \frac{1}{2k+1} - \ln\left(\frac{\alpha}{2}\right)\right\}\left(\frac{\alpha}{2}\right)^{2k+1} \tag{3.2}$$

We will show how (3.2) is obtained. The representation of $f_0(\alpha)$ in terms of a Meyer function is well known; this is expressed by existing parameters in terms of an integral containing the Γ-function [11]

$$f_0(\alpha) = \frac{1}{2}G_{13}^{21}\left(\frac{\alpha^2}{4}\middle|\begin{matrix} & 1 & \\ \frac{1}{2}, & \frac{1}{2}, & 0\end{matrix}\right) = \frac{1}{4\pi i}\int_L \Gamma^2\left(\frac{1}{2}-s\right)\left(\frac{\alpha}{2}\right)^{2s}\frac{ds}{s} \tag{3.3}$$

where L is the contour enclosing the singular points $s = 1/2, \ldots, 1/2 + k, \ldots$. To obtain the residues $\gamma_{-1,k}$ of integrand (3.3), we need the coefficients $a_{n,k}$, $b_{n,k}$, $c_{n,k}$ of the lowest two terms of the expansions of $\Gamma(s)$, $\Gamma^2(s)$, $s^{-1}(\alpha/2)^{2s}$ in the neighbourhoods of these singular points.

The coefficients $a_{n,k}$ are known [12]

$$a_{-1,k} = \frac{(-1)^k}{k!}, \quad a_{0,k} = \frac{(-1)^k}{k!}\left\{\sum_{n=1}^{k}\frac{1}{n} - C\right\} \tag{3.4}$$

(C is Euler's constant). The coefficients $b_{n,k}$ are obtained by multiplying the corresponding $a_{n,k}$. The coefficients $c_{n,k}$ are taken from the Taylor series of the function $s^{-1}(\alpha/2)^{2s}$.

Multiplying the corresponding $b_{n,k}$ and $c_{n,k}$, we obtain $\gamma_{-1,k}$, while the integral $f_0(\alpha)$ takes the form [13]

$$f_0(\alpha) = \frac{1}{2}\sum_{k=0}^{\infty}\gamma_{-1,k}, \quad \gamma_{-1,k} = \frac{4}{(2k+1)(k!)^2}\left\{\sum_{n=1}^{k}\frac{1}{n} - C + \frac{1}{2k+1} - \ln\frac{\alpha}{2}\right\}\left(\frac{\alpha}{2}\right)^{2k+1}$$

We can similarly obtain expansions of the remaining integrals (2.4). For example

$$f_1(\alpha) = \alpha - \sum_{k=1}^{\infty}\frac{4}{k!(k-1)!(2k+1)}\left\{\sum_{n=1}^{k-1}\frac{1}{n} - C + \frac{1}{2k} + \frac{1}{2k+1} - \ln\frac{\alpha}{2}\right\}\left(\frac{\alpha}{2}\right)^{2k+1} \tag{3.5}$$

$$f_2(\alpha) = \frac{2}{3}\alpha - \frac{4}{9}\left(\frac{\alpha}{2}\right)^3 + \frac{2}{3}\sum_{k=2}^{\infty}\frac{4}{k!(k-2)!(2k+1)}\times$$

$$\times\left\{\sum_{n=1}^{k-2}\frac{1}{n} - C + \frac{1}{2k-1} + \frac{1}{2k(2k+1)} - \ln\frac{\alpha}{2}\right\}\left(\frac{\alpha}{2}\right)^{2k+1} \tag{3.6}$$

Hence, the expression for the normal displacement can be represented in the following form, which is more convenient for calculations

$$U_z(x,y,0)=\frac{1}{2\pi^2}\frac{P}{b}\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\int_0^\infty\left\{f_0(\alpha\alpha_1)-f_0(\alpha\alpha_2)+\xi\sum_{k=0}^N[f_{k+1}(\alpha\alpha_1)-f_{k+1}(\alpha\alpha_2)]\sigma^k\right\}\times$$

$$\times\left\{\alpha+\omega\alpha^2(\alpha^2-\delta^2)\left[f_0(\alpha)+\xi\sum_{k=0}^N f_{k+1}(\alpha)\right]\sigma^k\right\}^{-1}\cos\left(\frac{2x}{b}\alpha\right)d\alpha \tag{3.7}$$

$$\omega=\frac{1}{\pi}\left(\frac{2}{b}\right)^4\frac{B}{2}\frac{\lambda+2\mu}{\mu(\lambda+\mu)},\quad N=\text{const}$$

When obtaining the expression for the normal displacement under the beam ($y = 0$) one can use the oddness of the functions $f_k(\alpha)$.

4. COMPARISON OF THE SOLUTION WITH THE RESULTS OBTAINED PREVIOUSLY IN [8]

The following expression for the normal displacement of the surface of an elastic half-space under a point load, moving along the surface, in a moving system of coordinates is given in [8]

$$u_3(x,y,0)=\frac{Pk^2}{2\pi\mu}\frac{1}{r\Omega}\sin^2\left(\frac{y}{r}\right)\left[1-h^2\sin^2\left(\frac{y}{r}\right)\right]^{1/2}$$

$$\Omega=4\left[1-h^2\sin^2\left(\frac{y}{r}\right)\right]^{1/2}\left[1-k^2\sin^2\left(\frac{y}{r}\right)\right]^{1/2}-\left[2-k^2\sin^2\left(\frac{y}{r}\right)\right]^2$$

$$r^2=x^2+y^2$$

We calculated the normal displacement of a section of the surface $0 < x < 50$ m, $y = 1.36$ m using (3.7) and (4.1). Figure 1 shows the displacements calculated from these formulae in a moving system (curves 1 and 2, respectively), for the case when the material of the beam is the same as the material of the base and is close in its physical-mechanical characteristics to limestone [14]. We took the following values of the constants: $P = 10^4$ N, $c = 44.44$ m/s, $b = 2.7$ m and $J = 0.0027$ m^4. The solutions are practically identical as one moves away from the point of application of the load, but the maximum difference in the displacements (when $x = 0$) is ~30%. A comparison of the displacements along the same section for characteristics of the material of the half-space close to those of granite, slate and gneiss and a permanent material of the beam (sandstone) showed that these displacements resemble curve 1 and differ only in value.

It follows from the above that when the velocities of motion of the load are an order of magnitude smaller than the velocity of sound, the determination of the displacements in the elastic base is simplified considerably. The approximate solution obtained agrees asymptotically with the existing solution [8] as one moves away from the region of application of the load, when the materials of the beam and the half-space are the same.

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